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On the strong coupling expansion for a generating functional

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Abstract. The strong coupling expansion (SCE) for the generating functional of the Green functions in the $(\varphi^*\varphi)^2_d$ model is investigated by the method of Schwinger equations. No lattices are used. The problem of boundary conditions is discussed. The exact (in all orders of SCE) relation between the two-particle function and the propagator has been obtained. From this relation the equation for the propagator is obtained. In the lower order this equation coincides with the known gap equation. A modification of the second Legendre transformation in the SCE region is presented.

1. Introduction

Perturbation theory is, so far, the only universal method used to calculate Green functions in quantum field theory.

The majority of physical applications, however, require going beyond the framework of perturbation theory. In this respect, a considerable interest is excited by the problem which is, somehow, inverse to the perturbation theory. This problem is the limit of strong coupling, i.e. expansion of Green functions in negative power series of the coupling constant. The problem has repeatedly been investigated (Hori 1962, Kaiser 1976, Ward 1978, Bender *et al* 1979, 1981). The feature common to all these investigations is the use of the lattice approximation to calculate the non-Gaussian path integrals. However, it is far from being easy to pass over to the continuous limit. Such a transition causes difficulties which are hard to overcome even in the case of simple models. The formal expressions obtained contain, as a rule, singular coefficients like $\delta(0)$, whose presence makes the performance of the renormalization program much more complicated. Besides, the propagator of the particle $\Delta(x)$ in the finite order of SCE is a function localized at the point $x=0$:

$$\Delta^{(n)}(x)=P_{n-1}(\partial^2)\delta(x). \tag{1.1}$$

Here P_n means a polynomial. Equation (1.1) can hardly be interpreted as a propagator.

To obtain physically understandable Green functions, one has to sum over the SCE.

Let us illustrate this idea by a simple example. Consider the theory of a scalar field φ with the mass m and the quadratic interaction $\mathscr{L}_{\text{int}}=\mu^2\varphi^2$. For this exactly soluble model, the SCE i.e. the expansion in the inverse powers of μ^2, is non-trivial and has property (1.1). For the propagator

$$\Delta=\frac{1}{\mu^2}\Delta^{(1)}+\frac{1}{(\mu^2)^2}\Delta^{(2)}+\dots$$

in the nth order we have

$$\Delta^{(n)}(x)=(-m^2+\partial^2)^{n-1}\delta(x)$$

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(in the Euclidean metric). In any finite order of $1/\mu^2$, the propagator in this simple model is the function with support at the point $x=0$. However, the summation over all orders of $1/\mu^2$ gives the right answer

$$\Delta=\sum_{n=1}^{\infty}\frac{1}{(\mu^2)^n}\Delta^{(n)}=\frac{1}{\mu^2+m^2-\partial^2}$$

whose interpretation is transparent. So, it is natural to suppose that in non-trivial models, physically sensible results are obtained only after summing up the SCE. This type of summation may, in a sense, be advantageous against the summation of a perturbation series. The reasons are, first, that the SCE has a simpler combinatorial structure in virtue of its polynomiality with respect to the sources; second, this expansion may have better convergence properties (the perturbation series is, at most, asymptotic).

The subject of the present paper is the investigation of the SCE for the scalar field theory with quartic interaction without exploiting the lattice. Underlying our investigation is the method of the iterative solution of the Schwinger equations for the generating functional†. In the next section, the iterative scheme for the strong coupling is formulated, and its general properties are discussed. The central point in constructing the SCE on the basis of the Schwinger equations is the boundary conditions problem. The SCE, unlike the iterative scheme of perturbation theory, requires that the functional differential equations with order higher than one, be solved. As a consequence, there arises an urgent need to have additional boundary conditions (which, generally, seems to be typical for non-perturbative approximations). Therefore, in constructing the SCE on the basis of the Schwinger equation, the propagator Δ in each finite order of the SCE should be understood as a boundary condition (input). To define the propagator, it is obligatory to use an alternative boundary condition, which, in turn, requires that the summation of the SCE be done.

A distinctive feature of the SCE is its polynomiality in the sources. This allows derivation of the exact relation (i.e. summed over all orders of the SCE relation), connecting a two-particle Green function with the propagator. This relation will be obtained in section 3. Then, proceeding from this relation and the requirement for consistency with perturbation theory, we obtain an approximate equation for the propagator (section 4). This equation coincides with the known gap equation for the given model, although it has been derived in a different way and is based on the summation of the SCE.

Section 5 discusses the Legendre transformation of the generating functional. A way to modify the Legendre transformation in the strong coupling region is proposed. Within this modification, the strong coupling gap equation for the propagator is derived. This equation is exact in the limit $\lambda\to\infty$, contrary to the ordinary (weak coupling) gap equation, which is exact at $\lambda\to 0$.

As an illustration to the above constructions, section 6 considers the so-called single-mode approximation (Cant and Rivers 1980), or, equivalently, the theory in 0-dimensional space.

Throughout the paper, the accent is laid on the properties of the SCE. In this connection, the well known gap equation physics is not discussed here. We also ignore the contribution from topologically non-trivial configurations and the related effects.

† The problem of the determination of the SCE from the Schwinger equations was investigated also by Bender *et al* (1989).

2. Schwinger equations and strong coupling expansion

Consider the complex scalar field $\varphi(x)$ in the d-dimensional Euclidean space $(x \in E_d)$ with the self-action $\lambda/2(\varphi^*\varphi)^2$. The generating functional of the Green functions for the model is:

$$G[\eta] = N^{-1} \int \mathrm{D}\varphi\, \mathrm{D}\varphi^* \exp\left\{-\int \mathrm{d}x\, \mathrm{d}y [\varphi^*(x)(\eta_0(xy) + \eta(xy))\varphi(y)] - \frac{\lambda}{2} \int \mathrm{d}x (\varphi^*(x)\varphi(x))^2\right\}. \tag{2.1}$$

Here N is the constant specified by the normalization condition $G[0] = 1$. The kernel of the quadratic form of the free action is the inverse free propagator $\eta_0 = m^2 - \partial^2 \equiv \Delta_c^{-1}$. The derivatives of G with respect to the bilocal source η determine the Green functions:

(i) the one-particle Green function (propagator):

$$\Delta(x-y) = \langle \varphi(x)\varphi^*(y)\rangle = -\frac{\delta G}{\delta\eta(yx)}\bigg|_{\eta=0} \tag{2.2}$$

(ii) the two-particle (four-point) Green function:

$$\mathscr{F}(xx'|yy') = \langle \varphi(x)\varphi(x')\varphi^*(y)\varphi^*(y')\rangle = \frac{\delta^2 G}{\delta\eta(y'x')\delta\eta(yx)}\bigg|_{\eta=0} \tag{2.3}$$

etc.

The translation invariance of the path integration measure leads to the Schwinger equations for the generating functional G:

$$\delta(x-y)G + \int \mathrm{d}x_1 (\eta_0(xx_1) + \eta(xx_1)) \frac{\delta G}{\delta\eta(yx_1)} = \lambda \frac{\delta^2 G}{\delta\eta(xx)\delta\eta(yx)} \tag{2.4a}$$

$$\delta(x-y)G + \int \mathrm{d}x_1 \frac{\delta G}{\delta\eta(x_1x)} (\eta_0(x_1y) + \eta(x_1y)) = \lambda \frac{\delta^2 G}{\delta\eta(yy)\delta\eta(yx)}. \tag{2.4b}$$

Equation (2.4a) may be recast in the form

$$\int \mathrm{d}x_1 (\eta_0(xx_1) + \eta(xx_1)) \frac{\delta}{\delta\eta(yx_1)} \{\exp\{\mathrm{tr}\ln(1+\eta\eta_0^{-1})\}G\} = \lambda \frac{\delta^2 G}{\delta\eta(xx)\delta\eta(yx)} \exp\{\mathrm{tr}\ln(1+\eta\eta_0^{-1})\}. \tag{2.5}$$

Similarly, one can write down (2.4b). From (2.5) one can easily derive the expansion of G into the power series of λ, choosing for the zero-order approximation the functional

$$G_0 = \exp\{-\mathrm{tr}\ln(1+\eta\eta_0^{-1})\}. \tag{2.6}$$

Notice that the generating functional G has to simultaneously obey two equations—(2.4a) and (2.4b). This is not essential for the perturbation theory, but may turn out important in the non-perturbative approach. For example, the functional

$$\exp\left\{\frac{1}{2\lambda} \int \mathrm{d}x_1\, \mathrm{d}x_2\, \mathrm{d}x_3 (\eta_0(x_1x_2) + \eta(x_1x_2))(\eta_0(x_1x_3) + \eta(x_1x_3))\right\}$$

is the solution of (2.4a), but does not satisfy (2.4b).

In order to obtain the expansion of the generating functional (2.1) in inverse power series of the coupling constant, let us redefine the integration variables in the path integral: $\varphi \to \lambda^{-1/4}\varphi$, $\varphi^* \to \lambda^{-1/4}\varphi^*$, and expand in a series the exponent of the quadratic form

$$\int D\varphi\, D\varphi^* \exp\{-\lambda^{-1/2}\varphi^*(\eta_0+\eta)\varphi - \tfrac{1}{2}(\varphi^*\varphi)^2\}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\frac{1}{\lambda^{n/2}} \int D\varphi\, D\varphi^* (\varphi^*(\eta_0+\eta)\varphi)^n \exp\{-\tfrac{1}{2}(\varphi^*\varphi)^2\}. \tag{2.7}$$

Equation (2.7) describes the SCE of the generating functional G in the language of the path integral. Note the fact that the term of order $\lambda^{-n/2}$ in the SCE is a polynomial of degree n in the source η. This essentially distinguishes the SCE from the perturbation expansion. In the latter, each term of the expansion in λ (starting with equation (2.6)) is non-polynomial in η.

Turn now to the Schwinger equations (2.4). Expanding G in a power series of $\lambda^{-1/2}$ as:

$$G = G^{(0)} + \lambda^{-1/2}G^{(1)} + \lambda^{-1}G^{(2)} + \ldots \tag{2.8}$$

we get the iterative schem of the SCE

$$\frac{\delta^2 G^{(n)}}{\delta\eta(yx)\delta\eta(xx)} = \delta(x-y)G^{(n-2)} + \int dx_1 (\eta_0(xx_1)+\eta(xx_1)) \frac{\delta G^{(n-2)}}{\delta\eta(yx_1)}$$
$$\frac{\delta^2 G^{(n)}}{\delta\eta(yx)\delta\eta(yy)} = \delta(x-y)G^{(n-2)} + \int dx_1 \frac{\delta G^{(n-2)}}{\delta\eta(x_1 x)} (\eta_0(x_1 y)+\eta(x_1 y)). \tag{2.9}$$

Equations (2.9) have to be completed by boundary conditions. One such condition is the normalization condition

$$G[0] = 1 \tag{2.10}$$

whence it follows that

$$G^{(0)}[0] = 1 \qquad G^{(n)}[0] = 0 \qquad n > 0. \tag{2.11}$$

In the iterative scheme of the perturbation theory, described by (2.5), the boundary condition suffices to define completely the generating functional in any order of λ, because the corresponding equations are always first-order equations with respect to the functional derivatives. In the strong coupling scheme, however, one has to solve second-order equations. This is why one more boundary condition is required to be set. It would be natural to set the derivative which in our case is the simplest Green function, i.e. propagator (2.2). Therefore, to boundary conditions (2.11) one has to add the boundary condition

$$\left.\frac{\delta G^{(n)}}{\delta\eta(yx)}\right|_{\eta=0} = -\Delta^{(n)}(x-y) \tag{2.12}$$

with $\Delta^{(n)}$ standing for the nth term of the expansion of the propagator in $\lambda^{-1/2}$:

$$\Delta = \lambda^{-1/2}\Delta^{(1)} + \lambda^{-1}\Delta^{(2)} + \ldots \tag{2.13}$$

The requirement to introduce an additional boundary condition into the iterative scheme of the Schwinger equations makes one more serious distinction between the

SCE and perturbation theory. Therefore, in each order of $\lambda^{-1/2}$, $G^{(n)}$ is the functional of not only the source η, but of the propagator as well: $G^{(n)}=G^{(n)}[\eta, \Delta]$. The solution of the equations of the iterative scheme of the strong coupling defines the Green functions as the functionals of the propagator. Therefore, in the construction of the SCE on the basis of the Schwinger equations, the propagator Δ should be understood not as an output, but as a boundary condition (input). To define the propagator, one has either to calculate a non-Gaussian path integral on the basis of the lattice approximation, accompanying with are the above difficulties of passing to a continuous limit, or use instead some alternative boundary condition. This boundary condition may, for instance, be consistent with perturbation theory:

$$G \approx G_0$$

at $\eta \to \infty$. However, it is not possible to introduce such a boundary condition directly, since, as has already been stated, each term of the SCE is polynomial in η. The use of this (or similar) additional boundary condition becomes possible only after the completion of the summation of the SCE. The problem of boundary conditions is the key problem in constructing the SCE by the method of Schwinger equations (problems of this kind are, in general, typical of non-perturbative constructions, Cant and Rivers 1980). It is necessary to emphasize that a calculation of non-Gaussian path integrals solves the problem of unique SCE construction. This can be illustrated by a toy model of quantum field theory in 0-dimensional space (see later, section 6). Convergence of the integral plays the role of the additional boundary condition. In this sense, Schwinger equations (2.4) have less information in the non-perturbative region than the integral in (2.1).

Consider now the system of equations (2.9) in greater detail. As was stated above, the nth term of the iterative expansion (2.9) is a polynomial of degree n in η:

$$G^{(n)}=G_n^{(n)}+G_{n-1}^{(n)}+\ldots+G_1^{(n)}+G_0^{(n)}$$

with $G_k^{(n)}=O(\eta^k)$. In accordance with (2.11) and (2.12)

$$G_0^{(n)}=0 \qquad \frac{\delta G_1^{(n)}}{\delta\eta}=-\Delta^{(n)}$$

when $n>0$. At $k>1$, the monomial $G_k^{(n)}$ satisfies the equations

$$\frac{\delta^2 G_k^{(n)}}{\delta\eta^2}=G_{k-2}^{(n-2)}+\eta\frac{\delta G_{k-2}^{(n-2)}}{\delta\eta}+\eta_0\frac{\delta G_{k-1}^{(n-2)}}{\delta\eta} \tag{2.14}$$

with homogeneous boundary conditions.

The first two terms of expansion (2.8) are, factually, trivial. They both satisfy the homogeneous equations:

$$\frac{\delta^2 G^{(0)}}{\delta\eta(yx)\delta\eta(xx)}=\frac{\delta^2 G^{(0)}}{\delta\eta(yx)\delta\eta(yy)}=0$$

(the same equation is true for $G^{(1)}$). From the boundary condition (2.11) one finds that $G^{(0)}=1$. The $G^{(1)}$ is sought for in the form

$$G^{(1)}=G_1^{(1)}=C\int \mathrm{d}x_1\,\mathrm{d}x_2\,\eta(x_1x_2)\Delta_1(a_1x_1+a_2x_2).$$

The boundary condition (2.12) yields $C=-1$, $a_1=-a_2=-1$, and

$$G^{(1)}=-\mathrm{tr}\,\eta\Delta^{(1)}.$$

The first non-trivial term of expansion (2.8) is

$$G^{(2)}=G_2^{(2)}+G_1^{(2)}.$$

From the boundary condition (2.12) it follows that

$$G_1^{(2)}=-\mathrm{tr}\ \eta\Delta^{(2)}$$

(generally, in all orders $G_1^{(n)}=-\mathrm{tr}\ \eta\Delta_n$).

The term $G_2^{(2)}$ is the solution of the equations

$$\frac{\delta^2 G_2^{(2)}}{\delta\eta(yx)\delta\eta(xx)}=\frac{\delta^2 G_2^{(2)}}{\delta\eta(yx)\delta\eta(yy)}=\delta(x-y)$$

with homogeneous boundary conditions.

Solve now the general system of equations:

$$\frac{\delta^2 G_f}{\delta\eta(yx)\delta\eta(xx)}=\frac{\delta^2 G_f}{\delta\eta(yx)\delta\eta(yy)}=f(x-y) \tag{2.15}$$

with an arbitrary function $f(x)$.

The solution is looked for in the form:

$$G_f=C\int \mathrm{d}x_1\ldots\mathrm{d}x_4\ \eta(x_1x_2)\eta(x_3x_4)f(a_1x_1+a_2x_2+a_3x_3+a_4x_4). \tag{2.16}$$

From (2.15) one obtains the values for the indeterminate coefficients:

$$a_1=-a_2=a_3=-a_4=-1 \qquad C=\tfrac{1}{2}.$$

So, the solution for (2.15) will be:

$$G_f=\tfrac{1}{2}\int \mathrm{d}x_1\ldots\mathrm{d}x_4\ \eta(x_1x_2)\eta(x_3x_4)f(-x_1+x_2-x_3+x_4)+\bar{G}_f \tag{2.17}$$

where $\bar{G}_f$ is a general solution of the corresponding system of homogeneous equations.

3. Two-particle function

Solution (2.17) allows calculation of the two-particle Green function (2.3) in all orders of the power expansion of $\lambda^{-1/2}$. Indeed, to calculate the two-particle function, it suffices to know $G_2^{(n)}$ for any n. According to (2.14), $G_2^{(n)}$ at $n>2$ is the solution of the system of equations

$$\begin{aligned}\frac{\delta^2 G_2^{(n)}}{\delta\eta(yx)\delta\eta(xx)}&=\frac{\delta^2 G_2^{(n)}}{\delta\eta(yx)\delta\eta(yy)}\\&=-\int \mathrm{d}x_1\ \eta_0(xx_1)\Delta^{(n-2)}(x_1-y)\\&=-\int \mathrm{d}x_1\ \Delta^{(n-2)}(x-x_1)\eta_0(x_1y).\end{aligned} \tag{3.1}$$

This system of equations can be referred to type (2.15) and, hence, its solution is given by formula (2.17) with $f=-(m^2-\partial^2)\Delta^{(n-2)}$.

Bring in the functional

$$G_2 \equiv \sum_{n=2}^{\infty} \frac{1}{\lambda^{n/2}} G_2^{(n)}$$

which is nothing but a part of the generating functional G quadratic in η. From formulae (2.17) and (3.1) it is possible to calculate G_2:

$$G_2 = \frac{1}{\lambda} \int \mathrm{d}x_1 \ldots \mathrm{d}x_4\, \eta(x_1x_2)\eta(x_3x_4) g(-x_1+x_2-x_3+x_4) + \bar{G}_2$$

where

$$g(x) = \delta(x) - \sum_{n=1}^{\infty} \frac{1}{\lambda^{n/2}} (m^2 - \partial^2)\Delta^{(n)}(x)$$
$$= \delta(x) - (m^2 - \partial^2)\Delta(x)$$

and $\bar{G}_2 = \mathrm{O}(\eta^2)$ is a solution of the system of homogeneous equations

$$\frac{\delta^2 \bar{G}}{\delta\eta(yx)\delta\eta(xx)} = \frac{\delta^2 \bar{G}}{\delta\eta(yx)\delta\eta(yy)} = 0. \tag{3.2}$$

So, one comes to the exact expression for the two-particle function:

$$\mathscr{F} = \mathscr{F}_S + \bar{\mathscr{F}}$$
$$\mathscr{F}_S(xx'|yy') = \frac{1}{\lambda}[\delta(x-y+x'-y') - (m^2-\partial^2)\Delta(x-y+x'-y)] \tag{3.3}$$

(the index S here means that the function is extracted from the SCE of the Schwinger equations). The function $\bar{\mathscr{F}}$ here satisfies the conditions

$$\bar{\mathscr{F}}(xx|yx) = \bar{\mathscr{F}}(xy|yy) = 0. \tag{3.4}$$

Also, $\bar{\mathscr{F}}$ must satisfy the physical requirements of the Bose-symmetry and the translation invariance:

$$\bar{\mathscr{F}}(xx'|yy') = \bar{\mathscr{F}}(x'x|yy') = \bar{\mathscr{F}}(xx'|y'y)$$
$$\bar{\mathscr{F}}(x-z, x'-z|y-z, y'-z) = \bar{\mathscr{F}}(xx'|yy').$$

At $\lambda^{-1/2} \to \infty$ $(\lambda \to 0)$, expanding $\Delta = \Delta_c + \ldots$, one formally obtains, from (3.3), that $\mathscr{F}_S = \mathrm{O}_\lambda(1)$. But the formula for $\mathscr{F}_S$ yields nothing that would resemble an expansion of the perturbation theory. To obtain consistency with perturbation theory, one should take into account the non-trivial solutions of equations (3.4). Undoubtedly, there are such solutions. Thus, the solution consistent in the zeroth order of λ is

$$\bar{\mathscr{F}}_0(xx'|yy') = \Delta(x-y)\Delta(x'-y') + \Delta(x-y')\Delta(x'-y) - 2\Delta(0)\Delta(x-y+x'-y'). \tag{3.5}$$

4. Propagator

Formula (3.3) is exact in the sense that it has been summed over all orders of the SCE. Therefore, following the consideration of section 3, it turns out possible to impose, as an additional boundary condition, the requirement of consistency with perturbation theory. Then (3.3) may be treated as an approximate equation for the propagator Δ. Call the two-particle function

$$\mathscr{F}(\Delta)=\mathscr{F}_S(\Delta)+\bar{\mathscr{F}}(\Delta)$$

n-consistent with perturbation theory, provided

$$\mathscr{F}(\Delta)\approx\mathscr{F}_n \tag{4.1}$$

where $\mathscr{F}_n$ is the perturbative approximation of the two-particle function in the nth-order λ-perturbation theory. In the usual perturbation theory, $\mathscr{F}_n$ is the functional of the free propagator Δ_c:

$$\mathscr{F}_n=\mathscr{F}_n(\Delta_c,\lambda).$$

Clearly, however, in any finite order λ one can always pass from Δ_c to the total propagator Δ:

$$\mathscr{F}_n=\mathscr{F}_n[\Delta,\lambda]+O(\lambda^{n+1}). \tag{4.2}$$

Relation (4.1), where $\mathscr{F}_n=\mathscr{F}_n[\Delta]$, can be understood to be an equation for the propagator Δ. In leading order, in accordance with (3.5), we get that

$$\lambda^{-1}[\delta(x)-(m^2-\partial^2)\Delta(x)]=2\Delta(0)\Delta(x). \tag{4.3}$$

From (4.3) one can readily obtain the expansion of the propagator in degrees of $\lambda^{-n/2}$. The formal expression for the first term of the expansion will then be

$$\Delta^{(1)}(x)=(2\delta(0))^{-1/2}\delta(x).$$

In lattice regularization, $\delta(0)$ should be understood as a^{-d}, where a means the spacing of the lattice. The subsequent terms of the SCE can be established from the iterative scheme

$$-(m^2-\partial^2)\Delta^{(n)}(x)=2\sum_{k=1}^{n+1}\Delta^{(k)}(0)\Delta^{(n+2-k)}(x) \tag{4.4}$$

which shows that the nth term of the expansion contains the $(n-1)$th degree of the inverse free propagator $\Delta_c^{-1}=m^2-\partial^2$. A similar result is also obtained by direct calculations (Hori 1962, Castoldi and Schomblond 1978, Bender *et al* 1979), underlying which is the use of lattice regularization.

Also, there is no difficulty in finding the exact solution to equation (4.3). Going over to p-space, we get

$$\tilde{\Delta}(p)=\frac{1}{2\lambda\Delta(0)+m^2+p^2} \tag{4.5}$$

with the constant $\Delta(0)$ specified by the equation

$$\Delta(0)=\int\frac{dp}{(2\pi)^d}\frac{1}{2\lambda\Delta(0)+m^2+p^2}.$$

Therefore, the 0-consistent two-particle function determines the free propagator with the renormalized mass

$$m_{\text{ren}}^2=2\lambda\Delta(0)+m^2. \tag{4.6}$$

(This renormalization is infinite when $d\geqslant 2$.)

Equation (4.3) is, clearly, approximate for both the strong coupling region and for small λ. It can easily be recognized as the gap equation in the Hartree–Fock-like approximation for the given model. Therefore, the well known Hartree–Fock approximation throws a bridge between SCE and the perturbation theory.

5. Legendre transformation

The approximation for the propagator, derived in the previous section from (3.3), may be obtained in a different way, namely by considering the second Legendre transformation (De Dominicis 1962, Dahmen and Jona-Lasinio 1967, Vassilev and Kazansky 1972). (It would be more precise to call it a Legendre transformation with respect to a bilocal source, but we keep here to the conventional terminology.) Let us first pass from the generating functional G to its logarithm

$$Z = \ln G$$

and introduce the functional

$$v(\eta) = \frac{\delta Z}{\delta \eta} \qquad v(0) = -\Delta. \tag{5.1}$$

Relation (5.1) may be considered as an equation in the source η. If unambiguous solvability of relation (5.1) for η is assumed: $\eta = \eta(v)$, then v can be treated as a new functional variable, and so the generating functional of the second Legendre transformation may be introduced:

$$\Gamma(v) = Z - \frac{\delta Z}{\delta \eta}\eta = Z - \operatorname{tr} v\eta. \tag{5.2}$$

As can easily be shown

$$\frac{\delta \Gamma}{\delta v} = -\eta.$$

The Schwinger equation than is written down as

$$1 + (\eta + \eta_0)v = \lambda\left[\left(\frac{\delta \eta}{\delta v}\right)^{-1} + vv\right]. \tag{5.3}$$

In the leading order of perturbation theory of λ we have

$$\eta_{(0)} = -(m^2 - \partial^2 + v^{-1}). \tag{5.4}$$

Substituting (5.4) into the right-hand side of (5.3) and switching off the source, we obtain (4.3) for the propagator Δ. This is the traditional derivation of the gap equation (Vassilev 1976). In this derivation, the given equation is in no way related to the SCE (as has been done in the previous section). The analysis of section 4 reveals that the gap equation for the propagator is consistent with both the expansion of the perturbation theory and the SCE. When $\lambda \to 0$, the lower order of the perturbation theory is reproduced correctly. But at $\lambda \to \infty$, the agreement is qualitative, since the subsequent approximations are comparable in the order of magnitude with the leading one at large λ.

In this connection, let us try to construct a consistency scheme based on the SCE instead of perturbation theory. To this end, one will have to modify the Legendre transformation, since the functional v specified by formula (5.1) is unfit to serve as a new functional variable *ad hoc*. The SCE for the functional Z is as follows:

$$Z = -\lambda^{-1/2} \operatorname{tr} \eta\Delta^{(1)} + \lambda^{-1} Z^{(2)} + \dots$$

Let us introduce the functionals

$$Z_s = \lambda(Z + \operatorname{tr} \eta\Delta)$$

and

$$u(\eta)=\frac{\delta Z_s}{\delta\eta}=\lambda\left(\frac{\delta Z}{\delta\eta}+\Delta\right). \tag{5.5}$$

At $\lambda\to\infty$, the quantities Z_s and $u(\eta)$ are of order unity. The boundary condition for $u(\eta)$ is homogeneous:

$$u(0)=0.$$

Understanding now u as the new functional variable, and (5.5) as the equation specifying η, let us introduce the *generating functional of the Legendre transformation in the strong coupling region*:

$$\Gamma_s(u)=Z_s-\frac{\delta Z_s}{\delta\eta}\eta=Z_s-\operatorname{tr}u\eta. \tag{5.6}$$

The generating functional Γ_s obeys the following relation, which is completely analogous to the property of the ordinary Legendre transformation:

$$\frac{\delta\Gamma_s}{\delta u}=-\eta.$$

In terms of Γ_s and u, the Schwinger equation becomes

$$1+\lambda^{-1}(\eta+\eta_0)u-(\eta+\eta_0)\Delta=\left(\frac{\delta\eta}{\delta u}\right)^{-1}+\lambda^{-1}uu+\Delta u+u\Delta+\lambda\Delta\Delta. \tag{5.7}$$

In the leading order of the SCE we have

$$\left(\frac{\delta\eta^{(0)}}{\delta u}\right)^{-1}=1-\Delta^{(1)}\Delta^{(1)}. \tag{5.8}$$

Substituting (5.8) in (5.7) and switching off the source, we come to the equation

$$(m^2-\partial^2)\Delta(x)+\lambda\Delta(0)\Delta(x)=\Delta^{(1)}(0)\Delta^{(1)}(x) \tag{5.9}$$

which may naturally be called the strong coupling gap equation for the propagator. The leading term of the expansion of the propagator Δ at $\lambda\to\infty$ is

$$\Delta^{(1)}(x)=\alpha(\delta(0))^{-1/2}\delta(x) \tag{5.10}$$

with α being a constant. Note, the right-hand side of (5.9) does not contain highly singular coefficients like $\delta(0)$:

$$(m^2-\partial^2+\lambda\Delta(0))\Delta(x)=\alpha^2\delta(x). \tag{5.11}$$

The solution to (5.11) is the renormalized free propagator with the renormalized mass

$$\bar{m}^2_{\text{ren}}=\lambda\Delta(0)+m^2.$$

Here α plays the role of the renormalization constant of the wavefunction.

The approximation, described by (5.9), is, in a sense, dual with respect to the ordinary gap equation. While the latter reproduces the propagator for $\lambda\to 0$, equation (5.9) is satisfied at $\lambda\to\infty$.

6. Single-mode approximation

As an illustration to the above constructions, look at the one-mode approximation (or, equivalently, the theory with $d=0$). In this simplified model, integral (2.1) becomes an ordinary two-dimensional integral calculated easily. The result is ($\eta_0 \equiv m^2$ at $d=0$)

$$G(\eta)=\frac{I((\eta+m^2)/(2\lambda)^{1/2})}{I(m^2/(2\lambda)^{1/2})} \tag{6.1}$$

where

$$I(x)=\exp x^2 \operatorname{erfc} x.$$

The Schwinger equation for G is an ordinary differential second-order equation

$$G+(\eta+m^2)\frac{\mathrm{d}G}{\mathrm{d}\eta}=\lambda\frac{\mathrm{d}^2G}{\mathrm{d}\eta^2}. \tag{6.2}$$

The solution to this equation is unambiguously fixed by the boundary conditions

$$G(0)=1 \tag{6.3}$$

$$G(\infty)=0. \tag{6.4}$$

where (6.3) is the normalization condition. As in the general case, condition (6.4) cannot be used in any finite order of the SCE due to the polynomial character of this expansion. Therefore, in the SCE one should use an additional boundary condition like (2.12). So, we see that the problem of boundary conditions for SCE is vital even within this simplified model. As the 'propagator' in this model we have

$$\Delta=-\frac{\mathrm{d}G}{\mathrm{d}\eta}\bigg|_{\eta=0}=\frac{(2/\pi\lambda)^{1/2}}{I(m^2/(2\lambda)^{1/2})}-\frac{m^2}{\lambda}. \tag{6.5}$$

When $\lambda\to\infty$

$$\Delta=\lambda^{-1/2}\Delta^{(1)}+\mathrm{O}(\lambda^{-1})$$

where

$$\Delta^{(1)}=(2/\pi)^{1/2}. \tag{6.6}$$

At $\lambda\to+0$

$$\Delta\to 1/m^2 \qquad \text{if } m^2>0$$

and

$$\Delta\approx -m^2/\lambda \qquad \text{if } m^2<0. \tag{6.7}$$

In the last case, the corrections are exponentially small.

Note, the iterative solution of (6.2) in the region of small λ will be $\Delta=m^{-2}+\mathrm{O}(\lambda)$, whatever the sign of m^2 might be. As the exact solution shows, for negative values of m^2 the iterative solution has nothing in common with the true behaviour of the propagator. This is an indication of an essential singularity at $\lambda=0$.

Consider now how the self-consistency approximations of the previous section within this simplified model. The ordinary gap equation (we call it the weak coupling gap equation and denote its solution as Δ_w) in this model looks like

$$1-m^2\Delta_w=2\lambda\Delta_w^2. \tag{6.8}$$

Due to the meaning of the weak coupling consistency, out of two possible solutions of this equation we should choose the one tending to the iterative solution $1/m^2$ at $\lambda \to 0$.

(i) $m^2 > 0$. One of the solutions fulfills this condition. The ratio of this solution to the exact solution (see (6.5))

$$R_w(\lambda) = \Delta_w(\lambda)/\Delta(\lambda)$$

falls monotonously from 1 to $\sqrt{\pi/4} \simeq 0.886$, with λ varying between 0 and ∞.

(ii) $m^2 < 0$. One of the solutions of (6.8) tends to the iterative solution m^{-2} that is, however, qualitatively different from the behaviour of the exact solution (see (6.7)). At $\lambda \to \infty$ this solution acquires a wrong sign. The other solution at small λ behaves as $-m^2/2\lambda$, which means that qualitatively its behaviour coincides with that of the exact solution. However, it is half as large in magnitude. When $\lambda \to \infty$, this solution has the right sign, and the ratio R_w tends to the same limit as in case (i) for the normal solution, although from below.

Turn now to consistency in the strong coupling region. Denoting the corresponding propagator as Δ_s, write for it the equation

$$m^2 \Delta_s + \lambda \Delta_s^2 = 2/\pi. \tag{6.9}$$

(i) $m^2 > 0$. One of the solutions tends to $\lambda^{-1/2}\Delta^{(1)}$ at $\lambda \to \infty$. In the interval $0 \leqslant \lambda < \infty$, the ratio

$$R_s(\lambda) = \Delta_s(\lambda)/\Delta(\lambda)$$

for this solution increases monotonously from $2/\pi \simeq 0.637$ to 1.

(ii) $m^2 < 0$. This is the most interesting case. The solution, tending at $\lambda \to \infty$ to $\lambda^{-1/2}\Delta^{(1)}$, behaves as $-m^2/\lambda$ when $\lambda \to 0$, which means that for this solution $R_s \to 1$ both in the strong coupling region and at small λ. In this case, max $R_s(\lambda) \simeq 1.127$ (at $\lambda_{max} \simeq 0.556(m^2)^2$), i.e. this solution approximates the exact solution to within $\simeq 13\%$ in the whole range of λ and tends to it at both the asymptotic regions.

Clearly, it is risky to draw far-reaching conclusions for the 'true' field theory from the analysis of this toy model. The infinite mass renormalization conceals the quantitative effects for $d \geqslant 2$.

7. Conclusion

The analysis shows that, in order to obtain some results on the basis of the SCE one has to sum up this expansion. An example of this summation is the formula for the two-particle function derived in section 3. Proceeding from this formula, one can obtain the equations for the propagator that are, in fact, a boundary condition. However, this formula does not say much about the two-particle function itself. Clearly, to get some information about a four-point function, the corresponding formulae for multi-particle functions are required. As is known, the Hartree–Fock approximation for the two-particle function is a sum of chain diagrams. Further analysis is needed to establish the relationship between the Hartree–Fock approximation and SCE for the two-particle function. In this connection, it seems interesting to make further investigations of the Legendre transformation from the point of view of the strong coupling expansion.

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